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Summary

The method used by Drazin to investigate the stability of unbounded, viscous, homogeneous, parallel shear flow to small wavenumber disturbances is extended to study the effect of thermal stratification on the stability of un bounded jets and shear layers. By this method the stability characteristics of continuous profiles are inferred from the stability characteristics of discontinuous profiles.

The differential equation which governs the stability of thermally stratified parallel shear flow of a viscous, heat-conducting, incompressible fluid is derived consistent with the Boussinesq approximation and the assumption of negligible viscous heating. It is shown that the solution of the governing differential equation in a layer of constant horizontal velocity and constant vertical temperature gradient can be expressed as a linear combination of exponential solutions. The matching conditions to be imposed upon the solutions in adjacent layers are derived.

The characteristic value problem for discontinuous jet and shear layers is posed by the requirement that the solutions of the governing differential equation satisfy the matching conditions and boundedness conditions for layers that extend to infinity. The analysis leads to a characteristic determinant which is required to vanish for the characteristic values of the parameters: the Reynolds number, the wavenumber and the wave speed. In order to find these characteristic values, an eigenvalue search routine is employed and curves of neutral stability for the shear layer and jet are found for several values of the Richardson number. The Prandtl number dependence is explored; but not completely. In particular, a Prandtl number dependence was found for the eigenvalues of the shear layer even in the absence of thermal stratification.

The stabilizing effect of the thermal stratification as parameterized by the Richardson number was found to be most stabilizing for small wavenumber (large-scale) disturbances. Moreover, since the model employing discontinuous profiles can only be used to infer the small wavenumber stability characteristics of continuous flows, no conclusions are drawn concerning a critical Richardson number which stabilizes the continuous flows. Nevertheless it is argued that the present approach represents a practical method of exploring the stability of thermally stratified shear flow of small-scale in the atmosphere as it occurs near regions of discontinuity in the vertical temperature gradient.

1. Introduction

A large part of hydrodynamic stability theory is devoted to the study of homogeneous parallel shear flow. For inviscid incompressible fluid the stability of such flows to small disturbances is governed by Rayleigh's equation,

$$(U(z) - c)\ (D^2 - \alpha^2)\ \phi(z) - (D^2U)\ \phi(z) = 0\ , \tag{1.1}$$

where $U(z)$ is the basic flow velocity, α is the disturbance wavenumber, c is the disturbance wave speed and $\phi(z)$ is the z dependent part of the disturbance stream function. The x-coordinate is taken as the direction of flow. The vertical coordinate, z, is directed across the flow and D represents $\frac{d}{dz}$. For viscous, incompressible fluid the stability of small disturbances is governed by the more general Orr-Sommerfeld equation,

$$(D^2 - \alpha^2)^2\ \phi(z) = i\alpha R\{(U - c)\ (D^2 - \alpha^2)\ \phi(z) - (D^2U)\ \phi(z)\}\ , \tag{1.2}$$

where R is the Reynolds number.

The stability analysis of the Orr-Sommerfeld equation involves the formulation of a characteristic value problem in which the solutions of the equation are required to satisfy certain boundary conditions. Eigenvalues and eigensolutions obtained from the solution of the characteristic value problem depend upon the basic velocity profile and the boundary conditions.

The major difficulty in obtaining the solution of the characteristic value problem for the stability analysis of homogeneous parallel shear flow is obtaining solutions to the Orr-Sommerfeld equation. At least three different approaches have been used. Each method has its limitations.

The classical method involves the asymptotic solution (for a review, see Reid[1]) of the Orr-Sommerfeld equation. It gives good results for large αR and is, therefore, well suited for stability computations of boundary-layer or channel flow. For such flows αR is usually larger than 100.

The second method is the numerical integration (for a review, see Betchov and Criminale[2]) of the Orr-Sommerfeld equation. It gives reasonable results for a wide range of αR but may lead to poor results if the Reynolds number is too large. There is, of course, a range of αR for which the results using the numerical analysis may be compared with the results using the asymptotic analysis. The agreement is good (see, for example, Lin[3] (p. 29)). The numerical integration of the Orr-Sommerfeld equation may require a fair amount of computer time. Especially, if many parameters are to be varied, such computations can become costly and it would be well to have another method which would require less computer time but still give good results for αR of order unity.

Drazin[4] has shown that it is possible to obtain the small wavenumber stability characteristics of unbounded flows, for which αR is known to be of order one, by considering the stability characteristics of discontinuous basic flows. In a layer of constant basic velocity the Orr-Sommerfeld equation has simple exponential solutions. Specifically Drazin demonstrated that the characteristic value problem for the discontinuous shear layer

$$U(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \end{cases} \tag{1.3}$$

leads to a curve of neutral stability, $R = 4\sqrt{3}\,\alpha$, to which the curves of neutral stability for continuous shear layer profiles are known to be asymptotic for small α (see, for example, Tatsumi and Gotoh[5] and Esch[6]).

Drazin also explored the stability characteristics of the discontinuous jet

$$U(z) = \begin{cases} 1 & |z| < 1 \\ 0 & |z| > 1 \end{cases} . \tag{1.4}$$

Figure 1 is the curve of neutral stability in the α,R plane and figure 2 is the corresponding curve in the α,c plane obtained from evaluating Drazin's equation (52). For comparison purposes several eigenvalues are given for the continuous Bickley jet $(U(z) = \mathrm{sech}^2\, z)$ after Clenshaw and Elliot[7] and Tatsumi and Gotoh. The agreement between Drazin's neutral curve for the discontinuous jet and the neutral curve for the Bickley jet is remarkable. This agreement is best, as expected, for small wavenumbers and becomes quite poor for $\alpha > 1$.

It seems reasonable, as suggested by Drazin, that the stability of discontinuous profiles in a stratified fluid may similarly be studied to investigate the stability of continuous unbounded stratified flow to disturbances of small wavenumbers.

2. The governing equations

The equations which govern the stability of thermally stratified shear flow in an incompressible, viscous, heat-conducting fluid are the Navier-Stokes equations, the continuity equation, a heat conduction equation, and the equation of state. With stable thermal stratification we anticipate two-dimensional waves as the most unstable disturbance and we, therefore, consider only the two-dimensional problem.

In a Cartesian coordinate system with x chosen as the direction of the unidirectional basic flow $U(z)$ and z as the vertical coordinate, the linearized equations are:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial z^2}\right) , \tag{2.1}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) - \frac{\rho'}{\rho} g ,$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 ,$$

$$\frac{\partial \theta'}{\partial t} + U\frac{\partial \theta'}{\partial x} + w'\frac{\partial \Theta}{\partial z} = \kappa\left(\frac{\partial^2 \theta'}{\partial x^2} + \frac{\partial^2 \theta'}{\partial z^2}\right) ,$$

and

$$\rho' = -\rho\gamma\theta' ,$$

where u' and w' are the longitudinal and vertical components of the perturbation velocity, p' is the perturbation pressure, ρ' is the perturbation density, and θ' is the perturbation temperature. The kinematic viscosity is ν , the thermometric conductivity is κ , and thermal expansion coefficient is γ . It has been assumed that heating due to viscous dissipation is negligible and the Boussinesq approximation has been made.

Consistent with the assumption that the most unstable disturbances are in the form of two-dimensional waves, periodic in x and t, we let

$$\begin{Bmatrix} u' \\ w' \\ p' \\ \theta' \end{Bmatrix} = \begin{Bmatrix} \hat{u}(z) \\ \hat{w}(z) \\ \hat{p}(z) \\ \hat{\theta}(z) \end{Bmatrix} \exp\{i\alpha(x-ct)\} . \tag{2.6}$$

If we now make the appropriate substitutions in equations (2.1-2.5) and use equation (2.5) in (2.2), we obtain

$$i\alpha(U - c)\hat{u} + \hat{w}DU = -\frac{i\alpha\hat{p}}{\rho} + \nu(D^2 - \alpha^2)\hat{u} , \tag{2.7}$$

$$i\alpha(U - c)\hat{w} = -\frac{D\hat{p}}{\rho} + \nu(D^2 - \alpha^2)\hat{w} + g\gamma\hat{\theta} , \tag{2.8}$$

$$i\alpha\hat{u} + D\hat{w} = 0 , \tag{2.9}$$

and

$$i\alpha(U - c)\hat{\theta} + \hat{w}D\Theta = \kappa(D^2 - \alpha^2)\hat{\theta} , \tag{2.10}$$

where D represents $\frac{d}{dz}$.

The continuity equation (2.9) provides the basis for the introduction of the perturbation stream function,

$$\psi' = \phi(z) \exp\{i(\alpha x - ct)\} ,$$

so that

$$u' = -D\psi' \quad \text{and} \quad w' = \frac{\partial \psi'}{\partial x} ,$$

or
$$\hat{u} = -D\phi \quad \text{and} \quad \hat{w} = i\alpha\phi .$$

Upon substitution of the perturbation stream function in equations (2.7) and (2.8), and elimination of the pressure, we obtain the fourth-order equation,

$$(U - c)\ (D^2 - \alpha^2)\phi - (D^2U)\phi = \frac{\nu}{i\alpha}\ (D^2 - \alpha^2)^2\ \phi + g\gamma\hat{\theta} . \tag{2.11}$$

If the last term on the right of this equation happens to be zero, it reduces to the dimensional form of the Orr-Sommerfeld equation. Equation (2.10) may now be expressed as

$$(U - c)\hat{\theta} + \phi D\Theta = \frac{\kappa}{i\alpha}\ (D^2 - \alpha^2)\hat{\theta} . \tag{2.12}$$

Taken together equations (2.11) and (2.12) are the sixth-order governing equations for the stability of thermally stratified parallel shear flow consistent with the Boussinesq approximation and the assumption of negligible viscous heating. The coupling between these equations is provided by the gravitational body force and the vertical gradient of the basic temperature.

If we now introduce a characteristic velocity U_* , a characteristic length L_* , and a characteristic temperature difference T_* , we obtain equations (2.11) and (2.12) in the non-dimensional form:

$$L_4\phi \equiv \{(i\alpha R)^{-1}\ (D^2 - \alpha^2)^2 - (U - c)\ (D^2 - \alpha^2) + D^2U\}\phi = Ri_b\ \hat{\theta} \tag{2.13}$$

and
$$L_2\hat{\theta} \equiv \{(i\alpha RPr)^{-1}\ (D^2 - \alpha^2) - (U - c)\}\hat{\theta} = -\ (D\Theta)\phi , \tag{2.14}$$

where $$R \equiv \frac{U_* L_*}{\nu} \quad , \quad Ri_b = \frac{g\gamma T_* L_*}{U_*^{\,2}} \quad , \text{ and } \quad Pr = \frac{\nu}{\kappa} \quad ,$$

are the Reynolds number, overall Richardson number, and Prandtl number respectively. By eliminating $\hat{\theta}$ from equations (2.13) and (2.14), we obtain the sixth-order equation

$$L_2 L_4 \phi + Ri_b (D\Theta)\phi = 0 \ . \tag{2.15}$$

Gage and Reid (1968)[8] and Gage (1971)[9] have studied the stability of thermally stratified viscous parallel flow in the presence of at least one rigid boundary by extending the usual asymptotic analysis for the Orr-Sommerfeld equation to treat the sixth-order equation. This method leads to good results for large αR but suffers from the limitation that only the special case of $Pr = 1$ can be treated. For unbounded flows we anticipate instability for small αR and we must therefore apply other methods to the stability analysis. As pointed out in the introduction, one such method has been used by Drazin[4] to study the stability of unbounded homogeneous jets and shear layers to small wavemunber disturbances by considering the stability of flows with piecewise constant basic velocity.

In order to generalize Drazin's method to stratified flows we first consider equation (2.15). Within a layer of constant basic velocity it can be reexpressed as

$$\begin{aligned} &\phi^{vi} - (\alpha^2 + \beta^2 + \gamma^2)\phi^{iv} + (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)\phi'' \\ &\quad - (\alpha^2\beta^2\gamma^2 - \alpha^2R^2 Pr\ Ri_b\ (D\Theta))\phi = 0 \end{aligned} \tag{2.16}$$

where $\beta^2 = \alpha^2 + i\alpha R(U - c)$ and $\gamma^2 = \alpha^2 + i\alpha R\ Pr(U - c)$.

Provided the basic temperature varies linearly with height, equation (2.16) is a sixth-order ordinary differential equation with constant coefficients. The solution is then a linear combination of exponential solutions of the form

$$\phi(z) = \kappa_1 e^{-a_1 z} + \kappa_2 e^{-a_2 z} + \kappa_3 e^{-a_3 z}$$

$$+ \kappa_4 e^{+a_1 z} + \kappa_5 e^{+a_2 z} + \kappa_6 e^{+a_3 z} \quad . \tag{2.17}$$

The a_i are roots of

$$a^6 - (\alpha^2 + \beta^2 + \gamma^2)a^4 + (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)a^2$$

$$- (\alpha^2\beta^2\gamma^2 - \alpha^2 R^2 \mathrm{Pr}\ \mathrm{Ri}_b(D\Theta)) = 0 \ . \tag{2.18}$$

If $\mathrm{Ri}_b(D\Theta) = 0$ and $\mathrm{Pr} \neq 1$, equation (2.16) can be expressed as

$$\{D^2 - \alpha^2\}\ \{D^2 - \beta^2\}\ \{D^2 - \gamma^2\}\phi = 0 \ , \tag{2.19}$$

and the solution is of the form

$$\phi(z) = \kappa_1 e^{-\alpha z} + \kappa_2 e^{-\beta z} + \kappa_3 e^{-\gamma z}$$

$$+ \kappa_4 e^{\alpha z} + \kappa_5 e^{\beta z} + \kappa_6 e^{\gamma z} \ . \tag{2.20}$$

If it should happen that $\mathrm{Ri}_b(D\Theta) = 0$ and $\mathrm{Pr} = 1$, the solution should be written in the form

$$\phi(z) = \kappa_1 e^{-\alpha z} + \kappa_2 e^{-\beta z} + \kappa_3 z e^{-\beta z}$$

$$+ \kappa_4 e^{\alpha z} + \kappa_5 e^{\beta z} + \kappa_6 z e^{\beta z} \ . \tag{2.21}$$

Finally, in the limit $Pr \to \infty$ with $Ri_b(D\theta) = 0$, equation (2.16) reduces to

$$\phi^{iv} - (\alpha^2 + \beta^2)\phi'' + \alpha^2\beta^2\phi = 0 \ , \tag{2.22}$$

which is the Orr-Sommerfeld equation with constant basic velocity considered by Drazin.

3. The Matching Conditions

The elements which go into the characteristic value problem for investigating the stability of thermally stratified flow are the solutions of the governing differential equations together with the simultaneous satisfaction of all boundary conditions. In developing the characteristic value problem for models with piecewise constant velocity profiles it is necessary to consider the proper matching conditions to be imposed on solutions in adjacent layers across a discontinuity in the basic flow. These matching conditions may be obtained by requiring continuity of certain physical quantities (viz. the perturbation velocity, stress and temperature) or by successively integrating the equation (2.15).

Drazin[4] develops the matching conditions for the Orr-Sommerfeld equation with a piecewise constant velocity profile by repeatedly integrating the Orr-Sommerfeld equation across the discontinuity. We may proceed in a similar fashion by repeatedly integrating equation (2.15) to show that

$$[\phi] = 0 \ , \tag{3.1}$$

$$[D\phi] = 0 \ , \tag{3.2}$$

$$[D^2\phi + i\alpha R(U - c)\phi] = 0 \ , \tag{3.3}$$

$$[D^3\phi - i\alpha R(U - c)D\phi] = 0 \ , \tag{3.4}$$

$$[D^4\phi - 3\alpha^2 D^2\phi - i\alpha R(U - c)D^2\phi] = 0 \ , \tag{3.5}$$

and

$$[D^5\phi - 3\alpha^2 D^3\phi - i\alpha R(U - c) \ \{D^3\phi - 2\alpha^2 D\phi\}] = 0 \ , \tag{3.6}$$

where the square brackets denote the jump, or difference of the quanity across the discontinuity. The first four matching conditions above were given by Drazin for the Orr-Sommerfeld problem.

Successive integration of the sixth-order equation is rather cumbersome. We may, however, illustrate the derivation of the matching conditions in the following manner. First consider equation (2.15) in the form

$$L_2 L_4 \phi = \{(i\alpha RPr)^{-1}(D^2 - \alpha^2) - (U - c)\} L_4 \phi = - Ri_b \Theta' \phi$$

and anticipate the continuity of ϕ , and $D\phi$. Integrating across the discontinuity at $z = z_o$, say, we find

$$\begin{aligned} \left[DL_4 \phi \right] &\equiv \lim_{\varepsilon \to o} \left[DL_4 \phi \right]_{z_o - \varepsilon}^{z_o + \varepsilon} \\ &= \lim_{\varepsilon \to o} \{(i\alpha RPr)^{-1} \alpha^2 + (U - c)\} \int_{z_o - \varepsilon}^{z_o + \varepsilon} L_4 \phi \, dz \\ &\quad - \lim_{\varepsilon \to o} Ri_b \int_{z_o - \varepsilon}^{z_o + \varepsilon} (D\Theta) \, \phi \, dz \, . \end{aligned} \tag{3.8}$$

Provided $L_4 \phi$ is bounded at z_o the first term on the right of equation (3.8) vanishes. The remaining term vanishes when Θ is continuous. If Θ is discontinuous at z_o , we integrate by parts to get

$$\lim_{\varepsilon \to o} - Ri_b \int_{z_o - \varepsilon}^{z_o + \varepsilon} \Theta' \phi \, dz = - Ri_b \left\{ [\Theta \phi] + \lim_{\varepsilon \to o} \int_{z_o - \varepsilon}^{z_o + \varepsilon} \Theta D \phi \, dz \right\} \tag{3.9}$$

and the matching condition will then be given by

$$\left[DL_4 \phi + Ri_b \, \Theta \phi \right] = 0 \, .$$

If we integrate again across the discontinuity, we find

$$\left[L_4\phi\right] = \lim_{\varepsilon\to 0} - Ri_b \int_{z_o-\varepsilon}^{z_o+\varepsilon} \Theta\phi \, dz = 0 \ .$$

Since the equation $L_4\phi = 0$ is the Orr-Sommerfeld equation, it follows that subsequent integrations must yield the same matching conditions as found by Drazin.

It is of some interest to consider what physical quantities are required to be continuous across the discontinuity in basic velocity consistent with the matching conditions derived above. The continuity of ϕ and $D\phi$ are consistent with the continuity of the normal and tangential perturbation velocities respectively. Equation (3.3) is associated with the continuity of vertical derivative of the normal perturbation stress,

$$\sigma_{zz}' = -\frac{p'}{\rho} + \frac{2}{R}\frac{\partial w'}{\partial z} \tag{3.12}$$

as it appears in

$$i\alpha(U - c)w' = -\frac{\rho' gL}{\rho U_o^2} + \frac{1}{R}\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right) + \frac{\partial}{\partial z}\left(-\frac{p'}{\rho} + \frac{2}{R}\frac{\partial w'}{\partial z}\right) \ , \tag{3.13}$$

which leads to the requirement

$$\left[D^2w' + \alpha^2 w' + i\alpha R(U - c)w'\right]$$
$$= \left[-\frac{\rho' gLR}{\rho U_o^2} + RD\sigma_{zz}'\right] = 0 \ . \tag{3.14}$$

Equation (3.14) reduces to (3.3) upon introduction of the stream function and application of (3.1). In a similar way the continuity of the normal perturbation stress

$$\sigma_{xx}' = -\frac{p'}{\rho} + \frac{2}{R}\frac{\partial u'}{\partial x} \tag{3.15}$$

as it appears in

$$i\alpha(U - c)u' = \frac{\partial}{\partial x}\left(-\frac{p'}{\rho} + \frac{2}{R}\frac{\partial u'}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right) \tag{3.16}$$

which leads to the requirement

$$\left[D^2u' - i\alpha R(U - c)u' + \alpha^2 u'\right] = \left[\frac{\partial}{\partial x}\ \sigma_{xx}'\right] = 0 \ . \tag{3.17}$$

Finally from equation (2.13) we observe that the continuity of θ' implies

$$\left[L_4\phi\right] = 0$$

which reduces to (3.5) and the continuity of $D\theta'$ implies

$$\left[DL_4\phi\right] = 0$$

which reduces to (3.6).

The usual physical quantities that one expects to be continuous in a viscous heat-conducting fluid are the perturbation velocity, the perturbation stress and the perturbation temperature. It seems reasonable to include the vertical derivative of the perturbation temperature for the present problem. The only discrepancy between the two approaches to the matching conditions appears in the third condition of equation (3.3). Esch[6], replaces (3.3) by the condition that perturbation shear stress is continuous, which implies

$$\left[D^2\phi + \alpha^2\phi\right] = 0 \ . \tag{3.18}$$

Drazin argues that in view of the aphysical nature of the present model the condition of equation (3.3) is more appropriate than (3.18). We have chosen to follow Drazin in this matter.

4. The Characteristic Value Problem

In Section 2 we derived the governing differential equation for the stability of two-dimensional disturbances in a layer of stratified fluid with constant basic velocity. By making use of the matching conditions to be imposed on solutions of this equation in adjacent layers we may now formulate the characteristic value problem.

In general the characteristic value problem is posed by requiring the solution (2.17) of eq. (2.16) to satisfy the matching conditions (3.1 -3.6) at each interface together with boundedness conditions for eigensolutions in regions which extend to infinity. We illustrate the technique below by posing the characteristic value problem for the simple shear layer with

$$U(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \end{cases} \quad \text{and} \quad \Theta(z) = z \ , \tag{4.1}$$

and the simple jet with

$$U(z) = \begin{cases} 1 & |z| < 1 \\ 0 & |z| > 1 \end{cases} \quad \text{and} \quad \Theta(z) = z \ . \tag{4.2}$$

4.1. The simple shear layer

The simple shear layer is a two layer-model. In region 1, $z > 0$, say, the solutions which remain bounded as $z \to \infty$ can be expressed as

$$\phi_1(z) = \kappa_{11} e^{-a_{11} z} + \kappa_{12} e^{-a_{12} z} + \kappa_{13} e^{-a_{13} z} \tag{4.3}$$

whereas in region 2 $z < 0$, the solutions which remain bounded as $z \to -\infty$ are

$$\phi_2(z) = \kappa_{21} e^{+a_{21} z} + \kappa_{22} e^{+a_{22} z} + \kappa_{23} e^{+a_{23} z} \ . \tag{4.4}$$

Application of the matching conditions at $z = 0$ leads to the algebraic equations

$$\kappa_{2j}(\delta_{1j} + \delta_{2j} + \delta_{3j}) = \kappa_{1j}(\delta_{1j} + \delta_{2j} + \delta_{3j}) \ , \tag{4.5}$$

$$a_{2j}\kappa_{2j} = - a_{1j}\kappa_{1j} \ , \tag{4.6}$$

$$\begin{aligned} &\{a_{1j}^2 + i\alpha R(1 - c)\ (\delta_{1j} + \delta_{2j} + \delta_{3j})\}\kappa_{1j} \\ &= \{a_{2j}^2 - i\alpha R(1 + c)\ (\delta_{1j} + \delta_{2j} + \delta_{3j})\}\kappa_{2j} \end{aligned} \tag{4.7}$$

$$\{a_{2j}^3 - i\alpha R(- 1 - c)a_{2j}\}\ \kappa_{2j} = \{- a_{1j}^3 + i\alpha R(1 - c)a_{1j}\}\ \kappa_{1j} \ , \tag{4.8}$$

$$\{a_{2j}^4 - 3\alpha^2 a_{2j}^2 + i\alpha R(1+c)a_{2j}^2\}\kappa_{2j} = \{a_{1j}^4 - 3\alpha^2 a_{1j}^2 - i\alpha R(1-c)a_{1j}^2\}\kappa_{1j} \ , \tag{4.9}$$

and

$$\begin{aligned} &\{a_{2j}^5 - 3\alpha^2 a_{2j}^3 + i\alpha R(1 + c)\ (a_{2j}^3 - 2\alpha^2 a_{2j})\}\ \kappa_{2j} \\ &= \{- a_{1j}^5 + 3\alpha^2 a_{1j}^3 - i\alpha R(1 - c)\ (- a_{1j}^3 + 2\alpha^2 a_{1j})\}\ \kappa_{1j} \ , \end{aligned} \tag{4.10}$$

where δ_{ij} is the Kronecker delta and summation from 1 to 3 is understood over repeated indices. Further rearrangement and simplification leads to the set of equations

$$\kappa_{1j}(\delta_{1j} + \delta_{2j} + \delta_{3j}) - \kappa_{2j}(\delta_{1j} + \delta_{2j} + \delta_{3j}) = 0 \ , \tag{4.11}$$

$$a_{1j}\ \kappa_{1j} + a_{2j}\ \kappa_{2j} = 0 \ , \tag{4.12}$$

$$\{ a_{1j}^2 + i\alpha R(1 - c)\ (\delta_{1j} + \delta_{2j} + \delta_{3j})\}\ \kappa_{1j}$$

$$- \{ a_{2j}^2 - i\alpha R(1 + c)\ (\delta_{1j} + \delta_{2j} + \delta_{3j})\}\ \kappa_{2j} = 0\ , \tag{4.13}$$

$$\{ a_{1j}^3 - i\alpha R(1 - c)a_{1j}\}\kappa_{1j} + \{ a_{2j}^3 - i\alpha R(- 1 - c)a_{2j}\}\kappa_{2j} = 0\ , \tag{4.14}$$

$$\{ a_{1j}^4 - 3\alpha^2 a_{1j}^2 - i\alpha R(1 - c)a_{1j}^2\}\ \kappa_{1j}$$

$$- \{ a_{2j}^4 - 3\alpha^2 a_{2j}^2 + i\alpha R(1 + c)a_{2j}^2\}\ \kappa_{2j} = 0\ , \tag{4.15}$$

and

$$\{ a_{1j}^5 - 3\alpha^2 a_{1j}^3 - i\alpha R(1 - c)\ (a_{1j}^3 - 2\alpha^2 a_{1j})\}\ \kappa_{1j}$$

$$+ \{ a_{2j}^5 - 3\alpha^2 a_{2j}^3 + i\alpha R(1 + c)\ (a_{2j}^3 - 2\alpha^2 a_{2j})\}\ \kappa_{2j} = 0\ . \tag{4.16}$$

Equations (4.11-4.16) form a system of six linear algebraic equations in terms of the six arbitrary constants κ_{11} , κ_{12} , κ_{13} , κ_{21} , κ_{22} and κ_{23} . The determinant of the coefficients of these constants must vanish for the existence of non-trivial eigensolutions. It follows that we may determine the eigenvalues of α , c , R , Pr and Ri_b by requiring that

$$\Delta(\alpha\ , c\ , R\ , Pr\ , Ri_b) = 0 \tag{4.17}$$

where Δ represents the determinant of the coefficients of the κ_{ij} in equations (4.11-4.16).

4.2. The discontinuous jet

The characteristic value problem for the discontinuous jet with basic velocity and temperature profile given by (4.2) is developed in a similar fashion as the simple shear layer treated above. The discontinuous jet is a three-layer model and matching conditions would appear to be required at the two interfaces $z = \pm 1$.

We consider first the solutions to the governing differential equation in the three layers: region 1 $(z > 1)$, region 2 $(-1 < z < 1)$, and region 3 $(z < -1)$. In region 1 , the solution is of the form

$$\phi_1(z) = \kappa_{11} e^{-a_{11} z} + \kappa_{12} e^{-a_{12} z} + \kappa_{13} e^{-a_{13} z} , \tag{4.18}$$

in region 2, it is of the form

$$\begin{aligned} \phi_2(z) = {} & \kappa_{21} e^{-a_{21} z} + \kappa_{22} e^{-a_{22} z} + \kappa_{23} e^{-a_{23} z} \\ & + \kappa_{24} e^{+a_{21} z} + \kappa_{25} e^{a_{22} z} + \kappa_{26} e^{a_{23} z} , \end{aligned} \tag{4.19}$$

and, finally,in region 3, it of the form

$$\phi_3(z) = \kappa_{31} e^{+a_{31} z} + \kappa_{32} e^{+a_{32} z} + \kappa_{33} e^{+a_{33} z} . \tag{4.20}$$

If we proceeded to apply the matching conditions at $z = \pm 1$ we would obtain a set of twelve homogeneous linear algebraic equations in the twelve unknown κ_{ij} . This procedure is unnecessary for the discontinuous jet with velocity, temperature gradient, and matching conditions symmetric about $z = 0$. It is then possible to consider even and odd eigensolutions separately and thereby reduce the order of the characteristic determinant.

It is known from previous investigations on the stability of parallel flow that the most unstable disturbances are associated with even ϕ . If we anticipate a similar result here, we may use the symmetry conditions to set

$$\kappa_{11} = \kappa_{31} \; , \; \kappa_{12} = \kappa_{32} \; , \; \kappa_{13} = \kappa_{33}$$

$$\kappa_{21} = \kappa_{24} \; , \; \kappa_{22} = \kappa_{25} \; , \text{ and } \kappa_{22} = \kappa_{26} \tag{4.21}$$

and we need only consider the region $z \geqslant 0$.

In region 1, the solution can be expressed in the simpler form

$$\phi_1(z) = \kappa_{11}'e^{-a_{11}(z-1)} + \kappa_{12}'e^{-a_{12}(z-1)} + \kappa_{13}'e^{-a_{13}(z-1)}$$

where $\kappa_{11}' = \kappa_{11}e^{-a_{11}}$, $\kappa_{12}' = \kappa_{12}e^{-a_{12}}$ and $\kappa_{13}' = \kappa_{13}e^{-a_{13}}$. (4.22)

In region 2, the solution can be written

$$\phi_2(z) = \kappa_{21}' \frac{\cosh a_{21}z}{\cosh a_{21}} + \kappa_{22}' \frac{\cosh a_{22}z}{\cosh a_{22}} + \kappa_{23}' \frac{\cosh a_{23}z}{\cosh a_{23}} \; , \tag{4.23}$$

where $\kappa_{21}' = \kappa_{21} \cosh a_{21}$, $\kappa_{22}' = \kappa_{22} \cosh a_{22}$ and $\kappa_{23}' = \kappa_{23} \cosh a_{23}$.

If we apply the matching conditions at $z = 1$ to these solutions, we obtain

$$\kappa_{1j}'(\delta_{1j} + \delta_{2j} + \delta_{3j}) - \kappa_{2j}'(\delta_{1j} + \delta_{2j} + \delta_{3j}) = 0 \tag{4.24}$$

$$a_{1j} \, \kappa_{1j}' + \{a_{2j} \tanh a_{2j}\} \, \kappa_{2j}' = 0 \tag{4.25}$$

$$a_{1j}^2\kappa_{1j}' - \{a_{2j}^2 + i\alpha R(\delta_{1j} + \delta_{2j} + \delta_{3j})\} \, \kappa_{2j}' = 0 \; , \tag{4.26}$$

$$a_{1j}^3 \kappa_{1j}' + \{a_{2j}^3 \tanh a_{2j} - i\alpha R a_{2j} \tanh a_{2j}\} \kappa_{2j}' = 0 , \quad (4.27$$

$$\{a_{1j}^4 - 3\alpha^2 a_{1j}^2 + i\alpha R c a_{1j}^2\} \kappa_{1j}'$$

$$- \{a_{2j}^4 - 3\alpha^2 a_{2j}^2 - i\alpha R(1 - c) a_{2j}^2\} \kappa_{2j}' = 0 , \quad (4.28)$$

and

$$\{a_{1j}^5 - 3\alpha^2 a_{1j}^3 + i\alpha Rc(a_{1j}^3 - 2\ {}^2 a_{1j})\} \kappa_{1j}'$$

$$+ \{a_{2j}^5 - 3\alpha^2 a_{2j}^3 - i\alpha R(1 - c)\ (a_{2j}^3 - 2\alpha^2 a_{2j})\}\ \tanh a_{2j} \kappa_{2j}' = 0 . \quad (4.29)$$

The characteristic equation for the stratified jet is obtained by requiring the determinant, Δ' , of κ_{ij}' to vanish in equations (4.24-4.29) and is of the form:

$$\Delta'(\alpha , c , R , Pr , Ri_b) = 0 .$$

5. Results of the computations

The elements of the characteristic value problem have been discussed in the preceding sections. In this section the procedure for the numerical solution of the characteristic value problem will be outlined and some typical results will be presented.

5.1. Numerical procedure

In order to evaluate the characteristic determinant we have used the Math Pack subroutine CGJR;a library subroutine supplied with the Univac 1108. The technique employed to find the eigenvalues is an application of Newton's method in two dimensions. To compute a curve of neutral stability we usually set the Richardson number and the Prandtl number equal to prescribed values. We may then pick a Reynolds number and search for the eigenvalues of α and c. The initial search is made by trial and error. When it is felt that the values of α and c are close enough to the desired eigenvalues, that is,when the value of the (complex) determinant is small enough, we use Newton's method to converge on the eigenvalues. The Reynolds number may be incremented and the old eigenvalues used to to initiate the eigenvalue search. In this way many points along the curve of neutral stability may be computed in a single run. Of course, we may just as conveniently increment α (or c) and search for eigenvalues of c (or α) and R. The Richardson number or the Prandtl number may also be incremented if desired (we then keep R, α, or c fixed and converge on the new eigenvalues of the other two). Finally, these computations are not restricted to curves of neutral stability. The imaginary part of c can be set equal to some non-zero value ($c_i > 0$ for instability and $c_i < 0$ for stability) and, in this way, growth rates may be computed.

5.2 Results for the discontinuous shear layer

Figures 3a, 3b, and 3c contain the results of the computations for the curves of neutral stability of the stratified discontinuous shear layer. Each figure contains curves of neutral stability in the α - R plane for three Richardson numbers but for the same Prandtl number. The Prandtl number differs for each figure.

Comparison of these three figures demonstrates the significance of the Prandtl number variation for this simple model. Figure 3a shows that the flow is considerably less stable for a Prandtl number of .1 than it is for a Prandtl number of 1 (figure 3b) or 10 (figure 3c). In all three figures the curves of neutral stability appear to be approximated by the intersection of two straight lines. The upper branch tends to a constant ratio R/α , depending on Pr , whereas the lower branch tends to a constant value of α , independent of the Prandtl number. Qualitatively each set of curves show similar behavior with the stabilizing effect of increasing the Richardson number most pronounced at small wavenumbers. Figure 4 demonstrates the effect of varying the Prandtl number for a point lying on the upper branch of the curve of neutral stability. As the Prandtl number gets large, the eigenvalues approach those of Drazin for $Ri_b = 0$. Figure 5 demonstrates s similar variation for a point lying on the lower branch of the curve of neutral stability. In each case the lower Prandtl number is associated with a destabilization of the shear flow.

5.3. Results for the discontinuous jet

Curves of neutral stability in the α , R plane for several values of the Richardson number and Prandtl number equal to unity are shown in Figure 6. For purposes of comparison the neutral curve obtained from equation (52) of Drazin[4] (1961) has been reproduced. Once again these curves show the stabilizing effect of thermal stratification to be most pronounced at small wavenumbers. The minimum critical Reynolds number increases from a value close to 4 and a wavenumber of .23 for $Ri = 0$ to a value of 15.6 and a corresponding wavenumber of .98 for $Ri = 0.0825$.

The corresponding curves of neutral stability in the α , c plane are shown in figure 7. At least for $Ri \geqslant 0.005$ the small wavenumber branches appear to approach limits as $R \to \infty : \alpha \to \alpha_s$, $c \to c_s = 0.5$ so that $Ri = \alpha_s^2 c_s^2$. The large wavenumber branch shows $\alpha \to \infty$, $c \to 0.5$ as $R \to \infty$ independent of the Richardson number. It is clear from these results that there is no critical Richardson number for which the flow is stabilized. However, none is expected because of the discontinuity in the velocity profile. The Prandtl number variation has yet to be studied in full. It is clear, however, that there is no appreciable Prandtl number variation for the homogeneous flow with $Ri_b = 0$.

6. Concluding Remarks

In this paper we have pursued a linear stability analysis of discontinuous jet and shear layer profiles in stratified, viscous, heat-conducting, and incompressible fluid. We have made the usual Boussinesq approximation in deriving the sixth-order governing differential equation. The Prandtl number dependence of this equation has been retained and we have demonstrated a destabilizing effect at small Prandtl numbers for the simple shear layer. This effect is also present without thermal stratification which suggests that the sixth-order equation (2.15) should, at least for some flows, be used instead of the Orr-Sommerfeld equation for studying the stability of homogeneous parallel shear flow.

The results of the stability analysis for the stratified shear layer and jet demonstrate the stabilizing effect of thermal stratification. This stabilizing effect, clearly illustrated in Figure 3 and Figure 6, is most pronounced for small wavenumber disturbances. Gage and Reid[8] and Gage[9] found a qualitatively similar stabilizing effect of thermal stratification on bounded flows by using an asymptotic analysis of the governing differential equation.

The inference of the stabilizing effect of thermal stratification on continuous flows from the stability analysis of discontinuous flows presented here is reasonable for small wavenumber disturbances in view of Drazin's work for homogeneous flows. We expect, that the lower branch of the curves of neutral stability as presented here for the discontinuous jet and shear layer approximate the corresponding curves for continuous flows over a limited range of Richardson numbers bounded below by zero. The behavior of the upper branch at large wavenumbers with $R \to \infty$ as $\alpha \to \infty$ is not surprising for a discontinuous flow. A continuous flow may be expected to approach a limiting value $\alpha \to \alpha_s$ as $R \to \infty$ in view of the known results for the homogeneous shear layer $U(z) = \tanh z$. A recent numerical study by Maslowe and Thompson[10] of the stratified shear layer, $U(z) = 1 + \tanh z$, and $\bar{\rho}(z) = \exp(- B \tanh z)$, confirms these expectations and demonstrates complete stabilization of the shear flow when the overall Richardson number exceeds 1/4 . We can make no predictions concerning critical Richardson numbers for continuous flows on the basis of our stability analysis of the discontinuous flows. The fact that we find no critical Richardson number for complete stabilization of the discontinuous flows is not surprising. After all the overall Richardson number may be greater than 1/4 while the local Richardson number is less than 1/4 .

The results presented here provide a qualitative framework in which to develop the more detailed numerical computations for continuous profiles. As mentioned above, such computations have already been made for a stratified shear layer by Maslowe and Thompson.

These results may have a wider application. In the atmosphere layers with nearly adiabatic lapse rates are often bounded above by very stable layers. the transition between these layers may be quite sharp and beyond the resolving capabilities of operational sounding devices. Often considerable shear develops between these layers. Several authors (for a review, see Bretherton[11]) have conjectured recently on the role of "Kelvin-Helmholtz" instability in generating internal waves at the interface. In this connection I believe that by "Kelvin-Helmholtz" instability we really mean "local shear instability of a stably stratified shear flow" which is a much more general state of affairs than the classical Kelvin-Helmholtz model implies. In this connection we intend to generalize the present model to treat the stability of waves at an interface between lavers of differing temperature gradient, different horizontal velocities and perhaps even differing viscosities.

Acknowledgements

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References

1. W. H. Reid, In Basic Developments in Fluid Dynamics, Vol. 1, pp. 249-307 (ed. M. Holt) New York, Academic Press (1965).

2. R. Betchov and W. O. Criminale, Jr., The Stability of Parallel Flows, Academic Press, New York 1967.

3. C. C. Lin, The theory of hydrodynamic stability, (Cambridge University Press, London 1955).

4. P. G. Drazin, J. F. M. 10 (1961).

5. Tatsumi and Gotoh, J. F. M. 7 (1960)

6. R. E. Esch, J. F. M. 7, 433 (1960).

7. C. W. Clenshaw and D. Elliott, Q. J. Mech. and Applied Math. 13, 300 (1960).

8. K. S. Gage and W. H. Reid, J. F. M. 33, 1 (1968).

9. K. S. Gage, J. F. M. To appear (1971).

10. S. A. Maslowe and J. M. Thompson, Physics of Fluids. To appear (1971).

11. F. Bretherton, Bulletin of the American Meteorological Society 52, 1 (1971).

Captions for the Figures

Figure 1. The curve of neutral stability in the α , R plane for the discontinuous jet after Drazin. The open circles and crosses show the points on the curve of neutral stability for the Bickley jet after Clenshaw and Elliott and Tatsumi and Gotoh respectively.

Figure 2. The curve of neutral stability for the discontinuous jet in the α , c plane after Drazin. Open circles and crosses refer to points on the corresponding curve for the Bickley jet after Clenshaw and Elliott and Tatsumi and Gotoh respectively.

Figure 3. The curves of neutral stability in the α , Re plane for the thermally stratified discontinuous shear layer at several different Prandtl numbers (a) $Pr = .1$, (b) $Pr = 1$, (c) $Pr = 10$.

Figure 4. The Prandtl number variation of the wavenumber α for $R = 2.0$ and $Ri = 0.001$.

Figure 5. The Prandtl number variation of the wavenumber α for $R = 10.0$ and $Ri = 0.20$.

Figure 6. The curves of neutral stability for the thermally stratified discontinuous jet in the α , R plane for several values of Ri and at $Pr = 1$.

Figure 7. The curves of neutral stability for the thermally stratified discontinuous jet in the c , α plane for several values of Ri and at $Pr = 1$.

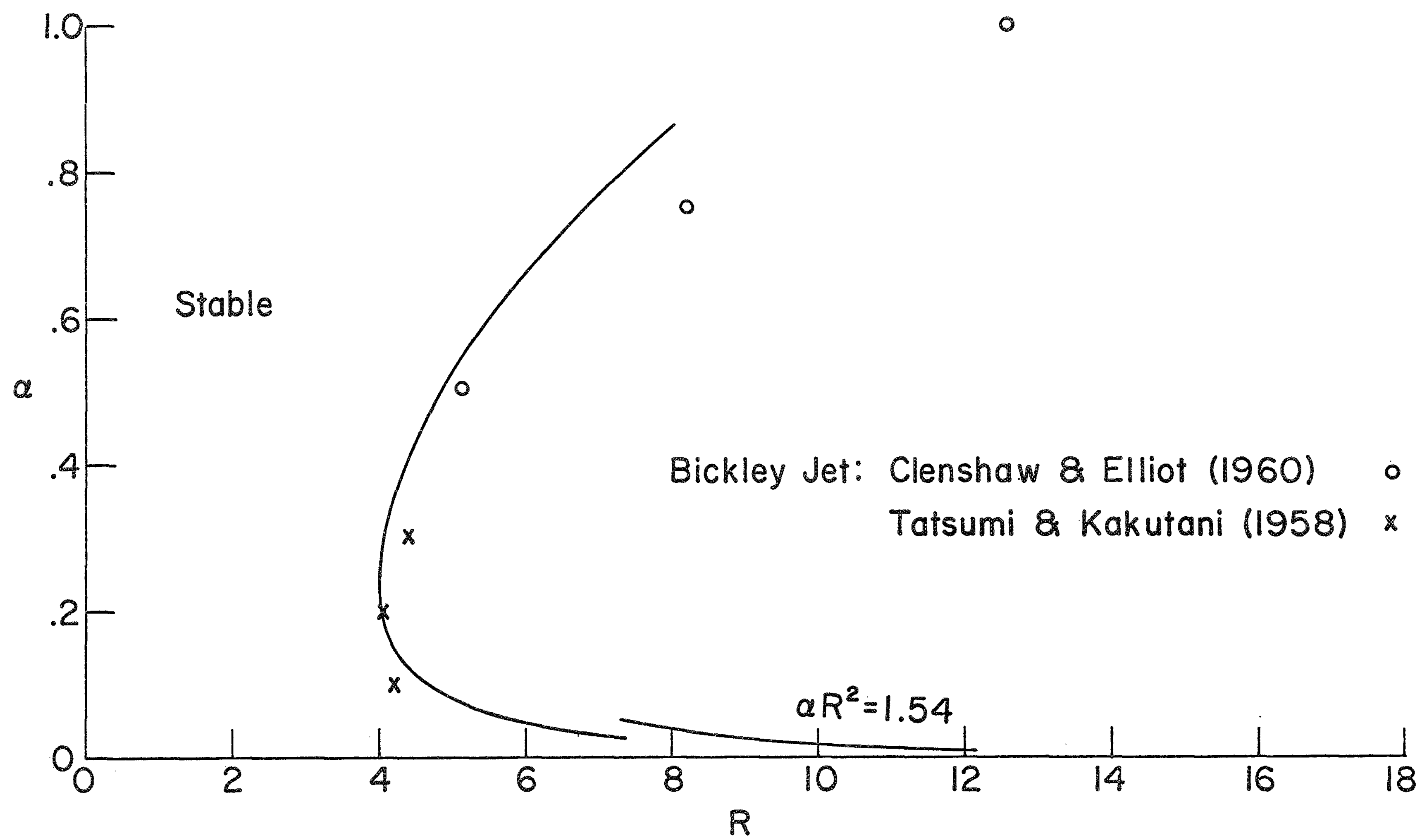
1.0
.8
.6
.4
.2
0
α
Stable
Bickley Jet: Clenshaw & Elliot (1960) o
Tatsumi & Kakutani (1958) x
αR²=1.54
0
2
4
6
8
10
12
14
16
18
R

Figure 1

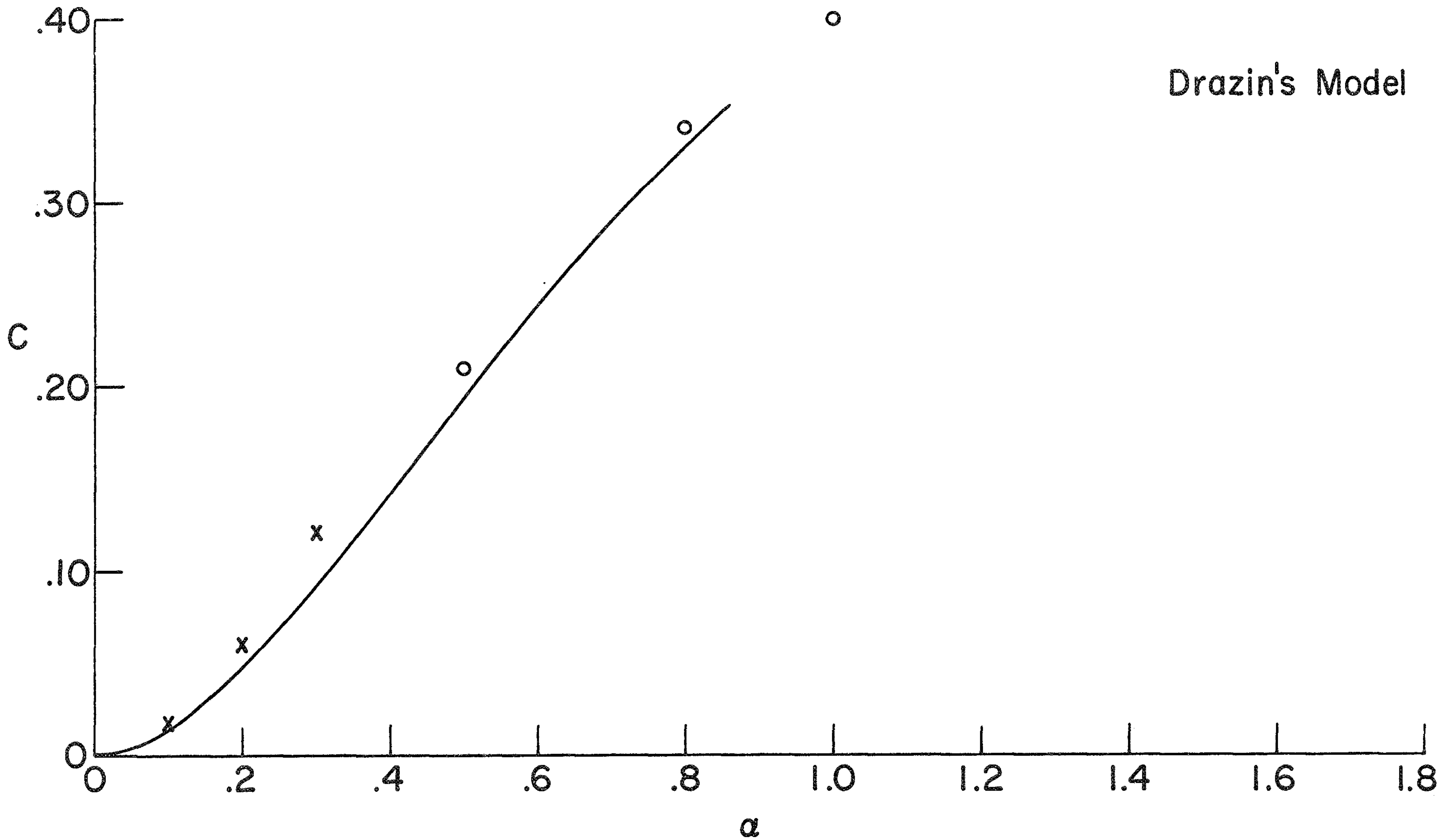

Figure 2

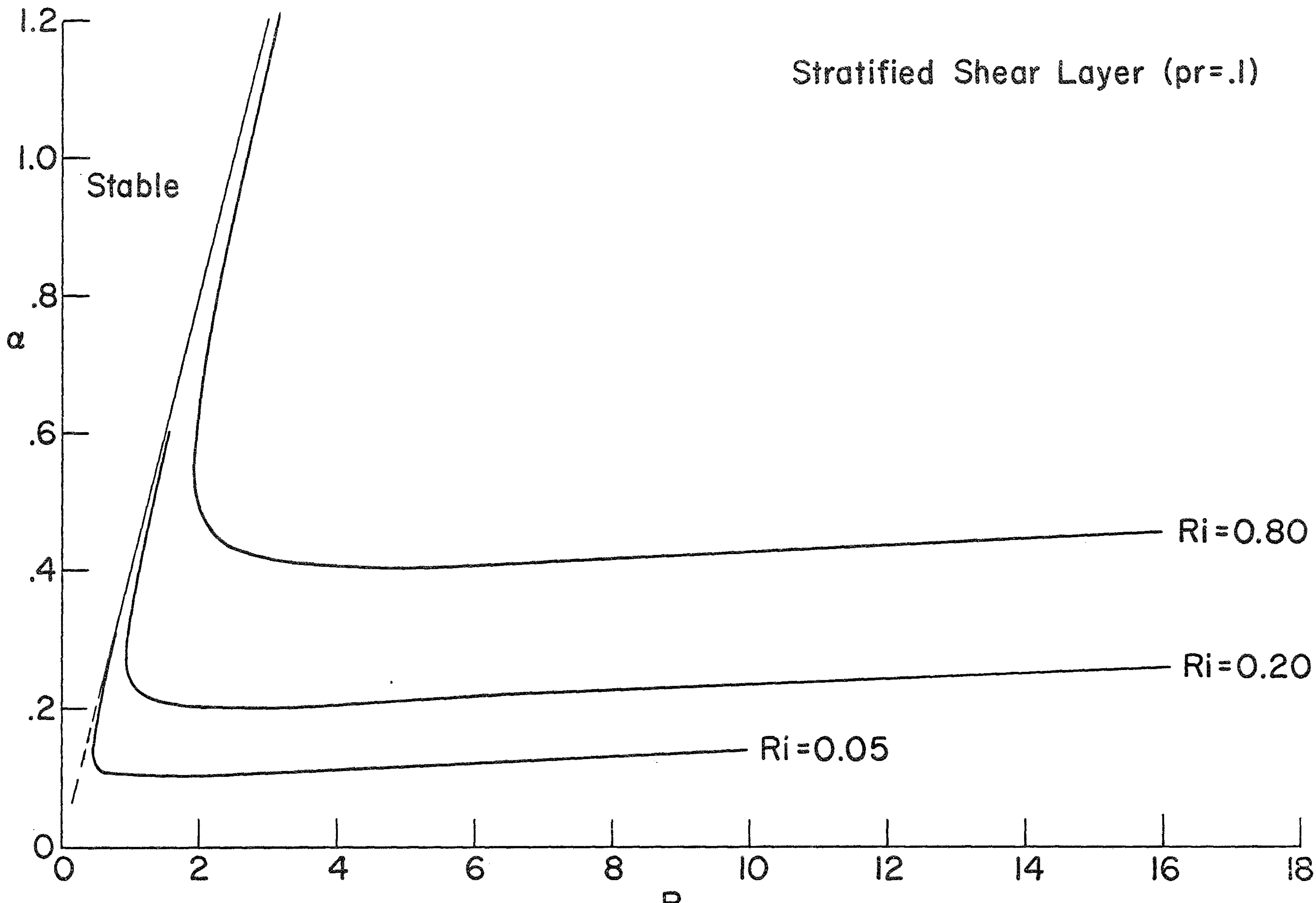
Stratified Shear Layer (pr=.1)
Stable
Ri=0.80
Ri=0.20
Ri=0.05
α
R
1.2
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
18

Figure 3 (a)

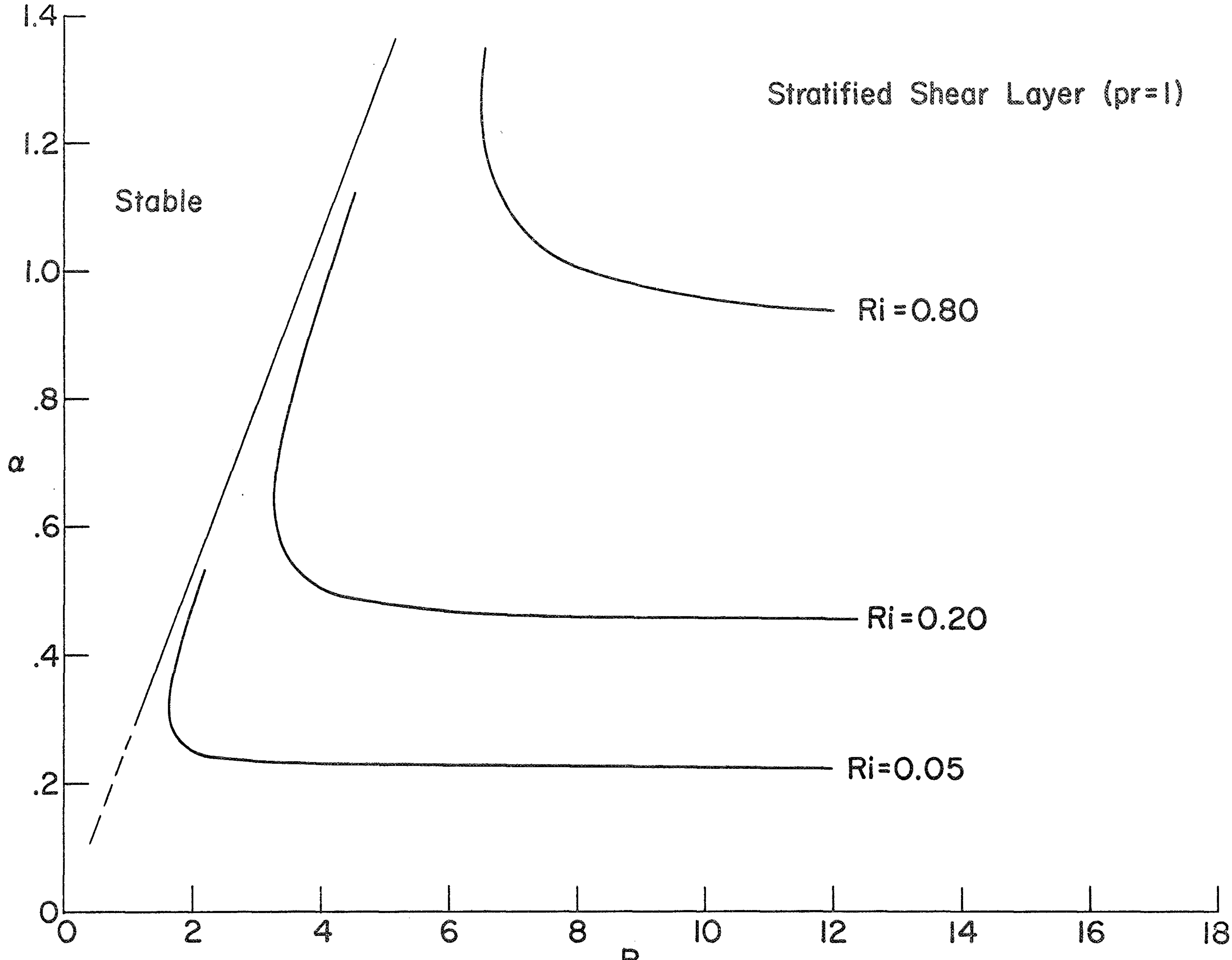
Stratified Shear Layer (pr=1)
Stable
Ri=0.80
Ri=0.20
Ri=0.05
1.4
1.2
1.0
.8
.6
.4
.2
0
α
0
2
4
6
8
10
12
14
16
18
R

Figure 3 (b)

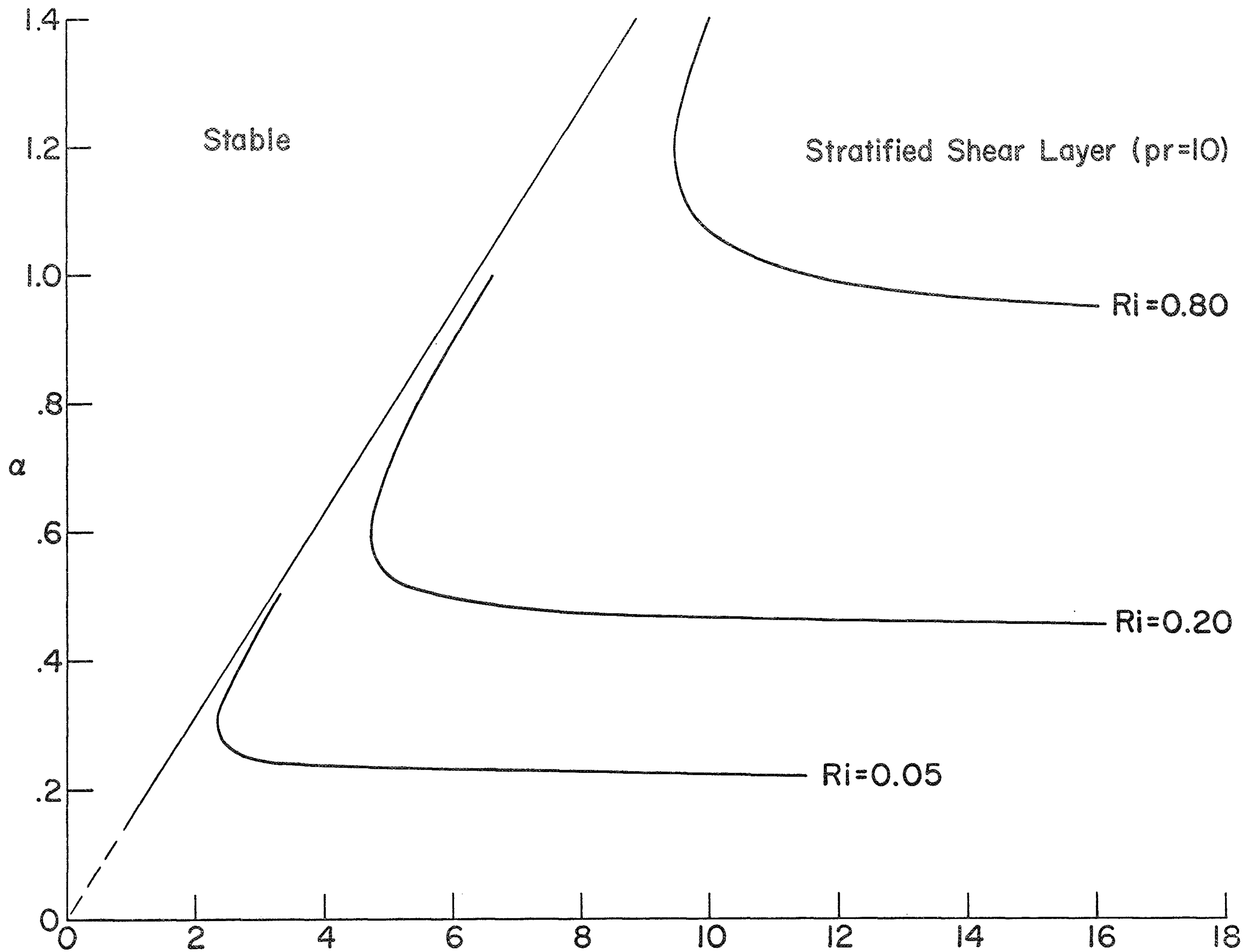
Stable
Stratified Shear Layer (pr=10)
Ri=0.80
Ri=0.20
Ri=0.05
1.4
1.2
1.0
.8
.6
.4
.2
0
α
0
2
4
6
8
10
12
14
16
18
R

Figure 3 (c)

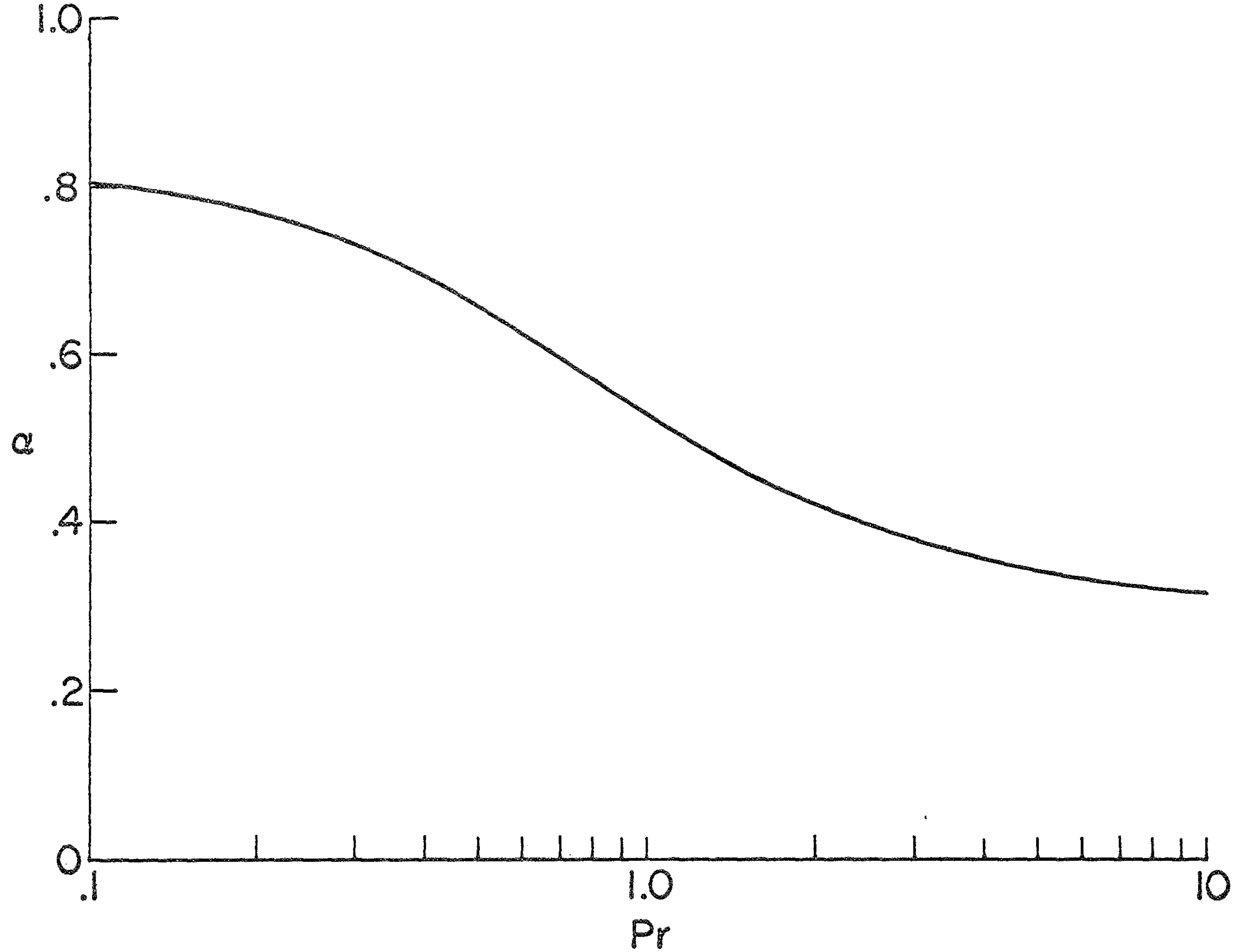

Figure 4

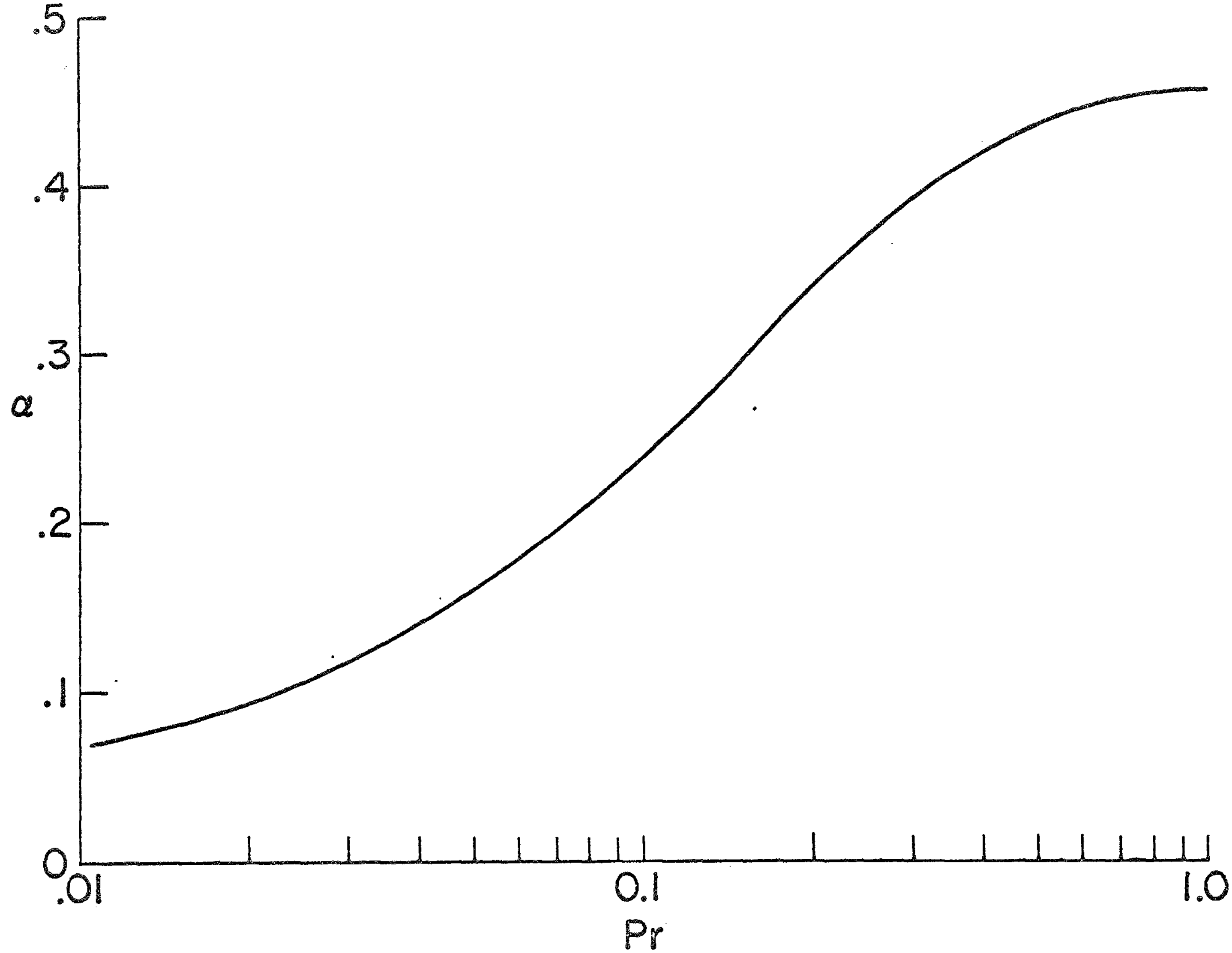

Figure 5

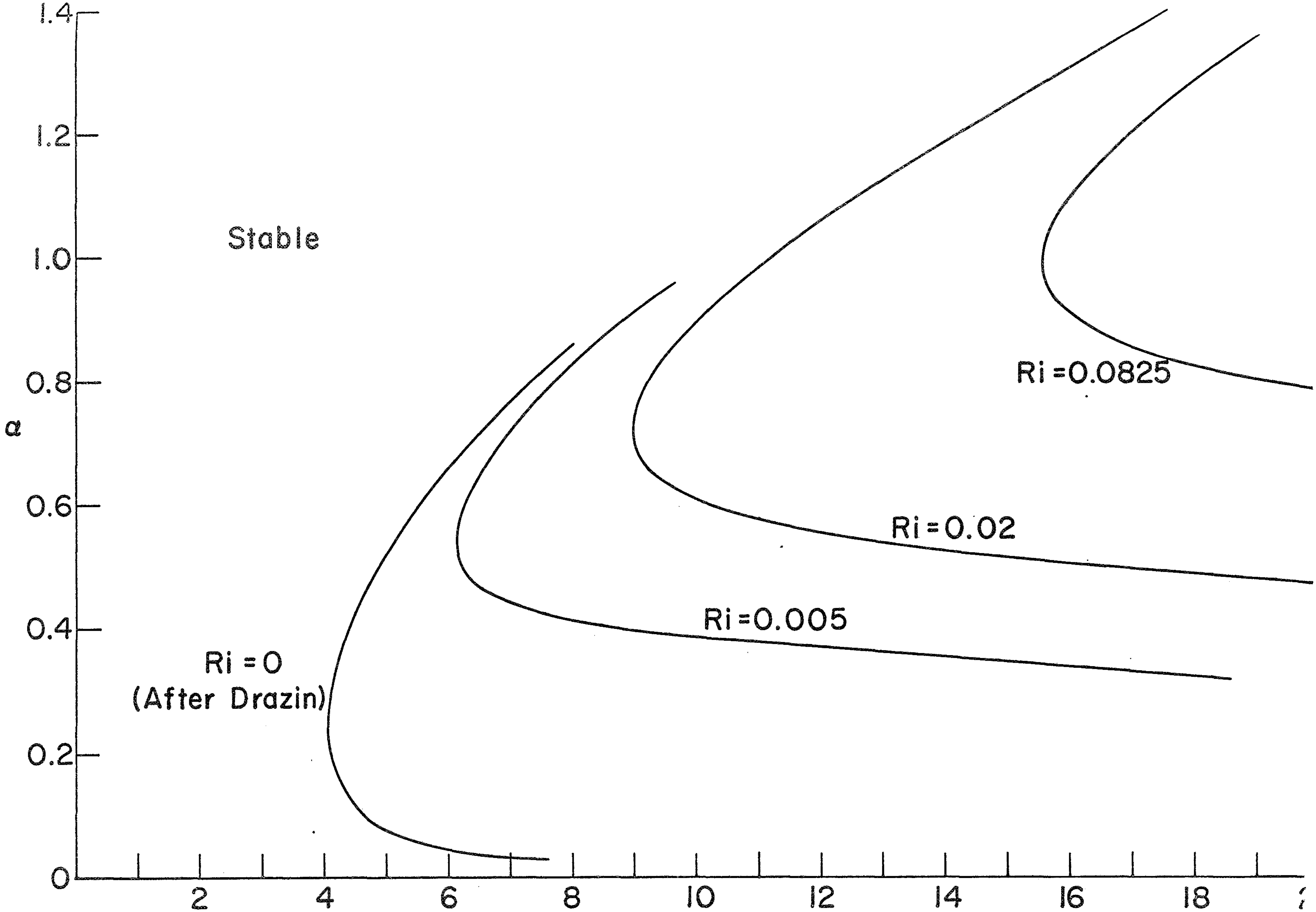
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
α
2
4
6
8
10
12
14
16
18
R
Stable
Ri = 0
(After Drazin)
Ri = 0.005
Ri = 0.02
Ri = 0.0825

Figure 6

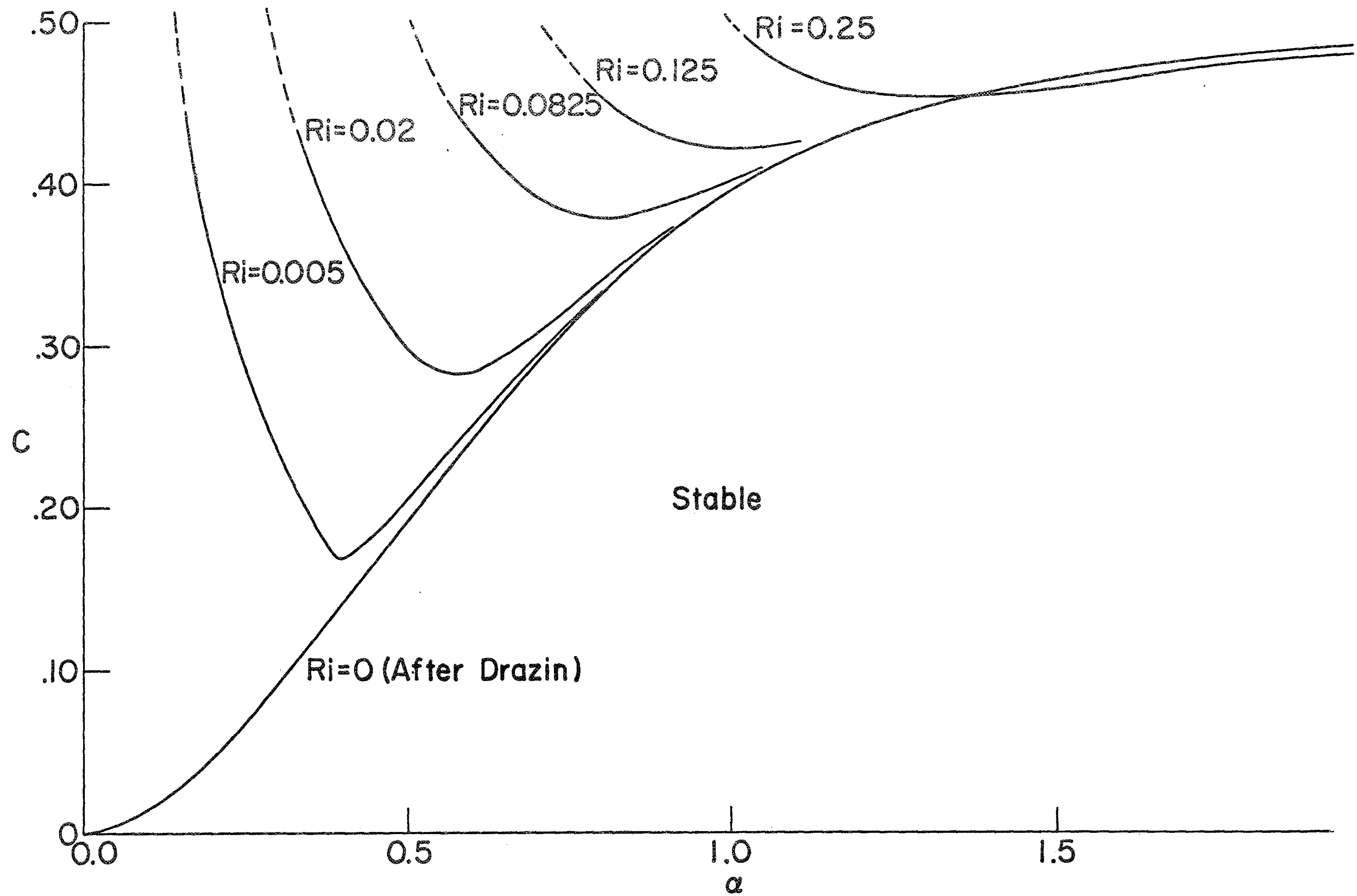
Ri=0.25
Ri=0.125
Ri=0.0825
Ri=0.02
Ri=0.005
Ri=0 (After Drazin)
Stable
C
.50
.40
.30
.20
.10
0
0.0
0.5
1.0
1.5
α

Figure 7